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On a Construction Related to the Non-abelian Tensor Square of a Group

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Abstract. Let G and G^{φ} be isomorphic groups. We introduce and study a quotient $\mathcal{V}(G)$ of the free product $G * G^{\varphi}$ which is a group extention of the non-abelian tensor square $G \otimes G$. This seems to bring computational advantages to calculate this last group. Looking over $\mathcal{V}$ as an operator in the class of groups we prove that it preserves properties of the argument G such as finiteness, set of prime divisors, nilpotency and solvability. For a finite p-group G we find a good polynomial bound for the order of $\mathcal{V}(G)$.

1. Introduction

The non-abelian tensor product $G \otimes H$ of the groups G and H, as introduced by R. Brown and J.-L. Loday [2], generalises the usual tensor product $\frac{G}{G'} \otimes_{\mathbb{Z}} \frac{H}{H'}$ of the abelianized groups, on the assumption that each of G and H acts on the other.

Specifically, given groups G, H each of which acts on the other (on the right)

$$G \times H \to G, (g,h) \mapsto g^h; H \times G \to H, (h,g) \mapsto h^g$$

in such a way that for all $g, g_1 \in G$ and $h, h_1 \in H$,

$$g^{h^{g_1}} = g^{g_1^{-1} h g_1} \quad \text{and} \quad h^{g^{h_1}} = h^{h_1^{-1} g h_1} \tag{1}$$

where G and H acts on itself by conjugation, then the *non-abelian tensor product* $G \otimes H$ is defined to be the group generated by all symbols $g \otimes h, g \in G, h \in H$, subject to the relations

$$g g_1 \otimes h = (g^{g_1} \otimes h^{g_1})(g_1 \otimes h) \tag{2}$$

$$g \otimes h h_1 = (g \otimes h_1)(g^{h_1} \otimes h^{h_1}) \tag{3}$$

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for all $g, g_1 \in G, h, h_1 \in H$, where the action of G on itself is the conjugation $g^{g_1} = g_1^{-1} g g_1$, and similarly for H.

In particular, as the conjugation action of a group G on itself satisfies (1), the *tensor square* $G \otimes G$ of a group G may always be defined. This tensor square is the focus of attention of [1] and [3], and constructions related to the general non-abelian tensor product are focused in [4].

The purpose of this article is to study a group which is also related to the above construction, defined as follows:

Let G and G^{φ} be isomorphic groups through $\varphi, g \mapsto g^{\varphi}, \forall g \in G$. We define the group

$$\mathcal{V}(G) := \langle G, G^{\varphi} \mid \ [g_1, g_2^{\varphi}]^{g_3} = [g_1^{g_3}, (g_2^{g_3})^{\varphi}] = [g_1, g_2^{\varphi}]^{g_3^{\varphi}}, \quad \forall g_1, g_2, g_3 \in G \rangle$$

(here we keep in mind that for elements h, k of any group, $h^k = k^{-1} h k$ and $[h, k] = h^{-1} h^k$).

Our motivation to introduce $\mathcal{V}(G)$ is that its *subgroup* $[G, G^{\varphi}]$ *is actually isomorphic to the non-abelian tensor square* $G \otimes G$ (Proposition 2.6).

Another construction related to $\mathcal{V}(G)$ is the one introduced by S. Sidki [10],

$$\chi(G) = \langle G, G^{\varphi} \mid \ [g, g^{\varphi}] = 1, \quad \text{for all} \quad g \in G \rangle,$$

which has, among other attributes, the property of being a finite group when G is finite. Considering the subgroup $\Delta(G)$ of $\mathcal{V}(G)$, generated by all $[g, g^{\varphi}], g \in G$, we obtain $\Delta(G) \leq \mathcal{V}(G)' \cap Z(\mathcal{V}(G))$. *The finiteness of* $\mathcal{V}(G)$ *then follows from the fact that* $\frac{\mathcal{V}(G)}{\Delta(G)}$ *is isomorphic to a certain natural factor of* $\chi(G)$ (Proposition 2.4).

By using techniques similar to those used in [5] and [9] we describe the lower central series and the derived series of $\mathcal{V}(G)$ in terms of the corresponding series of G. Our main results are the following:

Theorem A. *Let G be a nilpotent group of class c (resp. a solvable group of derived length ℓ). Then $\mathcal{V}(G)$ is a nilpotent group of class at most $c+1$ (resp. a solvable group of derived length at most $\ell + 1$).*

Theorem B. *Let G be a finite p-group of order p^n with G' of order p^m. Then $\mathcal{V}(G)$ is a p-group of order dividing $p^{n^2+2n-mn}$.*

In particular we obtain bounds for $G \otimes G$ similar to those of Jones [6] for the Schur Multiplier:

$$p^{d^2} \leq |G \otimes G| \leq p^{n(n-m)}$$

where $d = d(G)$ denotes the minimal number of generators of G.

2. Basic Results

In this section we derive some properties of the group $\mathcal{V}(G)$ and identify $G \otimes G$ as a subgroup of it. We use some standard commutator identities without reference (see, for instance, D. Robinson [8]):

For elements x, y, z in a group G, the conjugate of x by y is $x^y := y^{-1}xy$; the commutator of x and y is $[x, y] := x^{-1}x^y$ and our commutators are left normed, $[x, y, z] = [[x, y], z]$. The following identities hold:

$$\begin{aligned}
&[x, y] = [x, y^{-1}]^{-y} = [x^{-1}, y]^{-x};\\
&[xy, z] = [x, z]^y [y, z] = [x, z][x, z, y][y, z];\\
&[x, yz] = [x, z][x, y]^z = [x, z][x, y][x, y, z];\\
&[x, y^{-1}, z]^y [y, z^{-1}, x]^z [z, x^{-1}, y]^x = 1.
\end{aligned}$$

We simplify the definition of $\mathcal{V}(G)$ as

$$\mathcal{V}(G) = \langle G, G^{\varphi} \mid \ [g, h^{\varphi}]^{k^{\epsilon}} = [g^k, (h^k)^{\varphi}], \quad \text{for all} \quad g, h, k \in G, \epsilon \in \{1, \varphi\}\rangle,$$

where $\varphi: G \rightarrow G^{\varphi}, g \mapsto g^{\varphi}$ is a group isomorphism.

2.1 Lemma. *The following relations hold in* $\mathcal{V}(G)$*:*

(i) $[g_1, g_2^{\varphi}]^{[g_3, g_4^{\varphi}]} = [g_1, g_2^{\varphi}]^{[g_3, g_4]}, \quad \forall g_1, g_2, g_3, g_4 \in G;$

(ii) $[g_1, g_2^{\varphi}, g_3] = [g_1, g_2, g_3^{\varphi}] = [g_1, g_2^{\varphi}, g_3^{\varphi}]$ *and*
$[g_1^{\varphi}, g_2, g_3] = [g_1^{\varphi}, g_2, g_3^{\varphi}] = [g_1^{\varphi}, g_2^{\varphi}, g_3], \quad \forall g_1, g_2, g_3 \in G;$

(iii) $[g, g^{\varphi}]$ *is central in* $\mathcal{V}(G), \quad \forall g \in G;$

(iv) $[g_1, g_2^{\varphi}][g_2, g_1^{\varphi}]$ *is central in* $\mathcal{V}(G), \quad \forall g_1, g_2 \in G;$

(v) $[g, g^{\varphi}] = 1, \quad \forall g \in G'.$

Proof. (i) The defining relations of $\mathcal{V}(G)$ yield:

$$\begin{aligned}
[g_1, g_2^{\varphi}]^{[g_3, g_4^{\varphi}]} &= [g_1, g_2^{\varphi}]^{g_3^{-1} g_4^{-\varphi} g_3 g_4^{\varphi}} \\
&= [g_1^{g_3^{-1}}, (g_2^{g_3^{-1}})^{\varphi}]^{g_4^{-\varphi} g_3 g_4^{\varphi}} \\
&= \dots\dots\dots\dots\dots\dots \\
&= [g_1^{g_3^{-1} g_4^{-1} g_3 g_4}, (g^{g_3^{-1} g_4^{-1} g_3 g_4})^{\varphi}] \\
&= [g, g_2^{\varphi}]^{[g_3, g_4]};
\end{aligned}$$

(ii) From $[x, y] = x^{-1}x^{y}$ and commutator calculus we get

$$\begin{aligned}
[g_1, g_2, g_3^{\varphi}] &= [g_1^{-1} g_1^{g_2}, g_3^{\varphi}] \\
&= [g_1^{-1}, g_3^{\varphi}]^{g_1^{g_2}} \cdot [g_1^{g_2}, g_3^{\varphi}] \\
&= [g_1^{-1}, g_3^{\varphi}]^{g_2^{-1} g_1 g_2} [g_1, (g_3^{g_2^{-1}})^{\varphi}]^{g_2} \\
&\qquad \text{(by defining relations of } \mathcal{V}(G)) \\
&= [g_1, g_3^{\varphi}]^{-g_1^{-1} g_2^{-1} g_1 g_2} [g_1, (g_2 g_3 g_2^{-1})^{\varphi}]^{g_2} \\
&= [g_1, g_3^{\varphi}]^{-[g_1, g_2]} \cdot [g_1, (g_2^{-1})^{\varphi}]^{g_2} [g_1, (g_2 g_3)^{\varphi}] \\
&= [g_1, g_3^{\varphi}]^{-[g_1, g_2]} [g_1, g_2^{\varphi}]^{-1} [g_1, g_3^{\varphi}] [g_1, g_2^{\varphi}]^{g_3} \\
&= [g_1, g_3^{\varphi}]^{-[g_1, g_2^{\varphi}]} [g_1, g_2^{\varphi}]^{-1} [g_1, g_3^{\varphi}] [g_1, g_2^{\varphi}]^{g_3} \quad \text{(by (i))} \\
&= [g_1, g_2^{\varphi}]^{-1} [g_1, g_3^{\varphi}]^{-1} [g_1, g_3^{\varphi}] [g_1, g_2^{\varphi}]^{g_3} \\
&= [g_1, g_2^{\varphi}]^{-1} [g_1, g_2^{\varphi}]^{g_3} \\
&= [g_1, g_2^{\varphi}, g_3];
\end{aligned}$$

Now we observe that

$$\begin{aligned}
[g_1, g_2^{\varphi}, g_3^{\varphi}] &= [g_1 g_2^{\varphi}]^{-1} [g_1, g_2^{\varphi}]^{g_3^{\varphi}} \\
&= [g_1, g_2^{\varphi}]^{-1} [g_1, g_2^{\varphi}]^{g_3} \quad \text{(by defining relations)} \\
&= [g_1, g_2^{\varphi}, g_3]
\end{aligned}$$

The last two relations in (ii) follow by a symmetric argument.

(iii) It follows from (ii) that for all $g, h \in G$,

$$[g, g^{\varphi}, h] = [g, g, h^{\varphi}] = 1;$$

But

$$\begin{aligned}
[g, g^{\varphi}, h^{\varphi}] &= [g, g^{\varphi}]^{-1} \cdot [g, g^{\varphi}]^{h^{\varphi}} \\
&= [g, g^{\varphi}]^{-1} [g, g^{\varphi}]^{h} \\
&= [g, g^{\varphi}, h],
\end{aligned}$$

so that (iii) is proved:

(iv) For $g_1, g_2 \in G$ we get

$$\begin{aligned}[g_1g_2, (g_1g_2)^\varphi] &= [g_1, (g_1g_2)^\varphi]^{g_2}[g_2, (g_1g_2)^\varphi] \\ &= [g_1, g_2^\varphi]^{g_2}[g_1, g_1^\varphi]^{g_2^\varphi g_2}[g_2, g_2^\varphi][g_2, g_1^\varphi]^{g_2^\varphi} \\ &= [g_1, g_2^\varphi]^{g_2}[g_1, g_1^\varphi][g_2, g_2^\varphi][g_2, g_1^\varphi]^{g_2^\varphi} \quad \text{(by (iii))}\end{aligned}$$

Therefore, again by (iii), we can write

$$[g_1g_2, (g_1g_2)^\varphi][g_1, g_1^\varphi]^{-1}[g_2, g_2^\varphi]^{-1} = [g_1, g_2^\varphi]^{g_2}[g_2, g_1^\varphi]^{g_2^\varphi}$$

As the first member is central in $\mathcal{V}(G)$, on conjugating by $g_2^{-\varphi}$ and using the definition of $\mathcal{V}(G)$ we obtain

$$[g_1, g_2^\varphi][g_2, g_1^\varphi] = [g_1g_2, (g_1g_2)^\varphi][g_1, g_1^\varphi]^{-1}[g_2, g_2^\varphi]^{-1},$$

which belongs to the center of $\mathcal{V}(G)$;

As for (v), we first observe that when $g \in G'$ is a simple commutator, say $g = [x, y]$, then by (i) and (ii),

$$\begin{aligned}[[x, y], [x, y]^\varphi] &= [x, y, (x^{-1}x^y)^\varphi] \\ &= [x, y^\varphi, x^{-1}x^y] \\ &= [x, y^\varphi]^{-1}[x, y^\varphi]^{[x, y^\varphi]} \\ &= [x, y^\varphi]^{-1}[x, y^\varphi] = 1.\end{aligned}$$

Now for a general element $g \in G'$, say $g = [x_1, y_1] \dots [x_r, y_r]$, we use (i), (ii) and make induction on $r \geq 1$ to get

$$\begin{aligned}[g, g^\varphi] &= [[x_1, y_1] \dots [x_r, y_r], [x_1, y_1]^\varphi \dots [x_r, y_r^\varphi]] \\ &= \dots\dots\dots\dots\dots\dots\dots \\ &= [[x_1, y_1^\varphi] \dots [x_r, y_r^\varphi], [x_1, y_1^\varphi] \dots [x_r, y_r^\varphi]] = 1,\end{aligned}$$

proving (v). □

2.2 Lemma. *Let a, b, x be elements in G such that $[x, a] = 1 = [x, b]$. Then*

$$[a, b, x^\varphi] = 1 = [[a, b]^\varphi, x].$$

Proof. By Lemma 2.1 (ii) we obtain

$$\begin{aligned}[a,b,x^{\varphi}) &= [a,b^{\varphi},x]\\ &= [a,b^{\varphi}]^{-1}\cdot[a,b^{\varphi}]^{x}\\ &= [a,b^{\varphi}]^{-1}[a^{x},(b^{x})^{\varphi}]\\ &= [a,b^{\varphi}]^{-1}[a,b^{\varphi}] = 1.\end{aligned}$$

The other identity follows by the symmetry in part (ii) of Lemma 2.1. □

2.3 Lemma. *Let x, y be elements of G such that $[x,y]=1$. Then*

(i) $[x^n, y^{\varphi}] = [x, y^{\varphi}]^n = [x,(y^{\varphi})^n]$, *for all $n \in \mathbb{Z}$;*

(ii) *If x and y are torsion elements of orders $o(x)$ and $o(y)$, then $o([x,y^{\varphi}])$ divides the g.c.d.$(o(x), o(y))$.*

Proof. (i) is proved by induction for $n \geq 0$, while

$$[x,y^{\varphi}]^{-1} = [x^{-1},y^{\varphi}]^{x} = [x^{-1},(y^{x})^{\varphi}] = [x^{-1},y^{\varphi}];$$

(ii) is a consequence of (i). □

Remark 1. By the symmetry between the defining relations of $\mathcal{V}(G)$ we note that the isomorphism φ extends uniquely to an automorphism Ψ of $\mathcal{V}(G)$ sending $g \mapsto g^{\varphi}, g^{\varphi} \mapsto g$ and $[g_1, g_2^{\varphi}] \mapsto [g_2, g_1^{\varphi}]^{-1}$, for all $g, g_1, g_2 \in G$.

Remark 2. For a finite group G, we can get the finiteness of $\mathcal{V}(G)$ making use of the finiteness of the following group $\chi(G)$ (cf. S. Sidki [10]):

For the given isomorphic pair G, G^{φ}, consider the group

$$\chi(G) := \langle G, G^{\varphi} \mid [g, g^{\varphi}] = 1, \quad \forall g \in G\rangle.$$

Then we quote the following results [10] on $\chi(G)$ (see also [5,9]): "Let G be a finite π-group (π a set of primes), finite nilpotent or solvable of finite degree. Then $\chi(G)$ is also a finite π-group, finite nilpotent or solvable of finite degree". It should be noted that $\chi(G)$ has a subgroup $R(G)$ such that the relations $[g_1, g_2^{\varphi}]^{g_3^{\varphi}} = [g_1^{g_3}, (g_2^{g_3})^{\varphi}]$ hold in $\dfrac{\chi(G)}{R(G)}$ for all $g_1, g_2, g_3 \in G$ ([10], Lemma 4.11 (iii)). Here $R(G) = [G, L(G), G^{\varphi}]$, where $L(G)$ is given by $L(G) = [G, \varphi] := \langle g^{-1}g^{\varphi}, \ \forall g \in G\rangle$.

Returning to our group $\mathcal{V}(G)$ we note that on introducing the relations $[g, g^{\varphi}] = 1$ for all $g \in G$ it renders an epimorphism $\rho: \mathcal{V}(G) \to \dfrac{\chi(G)}{R(G)}$ defined by $g \mapsto gR(G), g^{\varphi} \mapsto g^{\varphi}R(G)$, $\forall g \in G$, $\forall g^{\varphi} \in G^{\varphi}$, whose Kernel $\Delta(G)$ is contained in $Z(\mathcal{V}(G)) \cap \mathcal{V}(G)'$, by Lemma 2.1 (iii). This implies that $\Delta(G)$ is a homomorphic image of the Schur Multiplier of $\dfrac{\chi(G)}{R(G)}$ which, together with the above quoted results, gives

2.4 Proposition. *Let G be a finite π-group (π a set of primes), finite nilpotent or solvable of finite degree. Then $\mathcal{V}(G)$ is also a finite π-group, finite nilpotent or solvable of finite degree.*

Let N be a normal subgroup of G. We set $\bar{G}$ for the quotient group $\dfrac{G}{N}$ and note that the canonical epimorphism $\pi = G \to \bar{G}$ gives raise to an epimorphism $\tilde{\pi}: \mathcal{V}(G) \to \mathcal{V}(\bar{G})$ such that $g \mapsto \bar{g}, g^{\varphi} \mapsto \overline{g^{\varphi}}$, where $\overline{G^{\varphi}} = \dfrac{G^{\varphi}}{N^{\varphi}}$ is identified with $\bar{G}^{\varphi}$.

2.5 Proposition. *With the above notation we have*

(i) $[N, G^{\varphi}] \trianglelefteq \mathcal{V}(G), [G, N^{\varphi}] \trianglelefteq \mathcal{V}(G)$;

(ii) $\operatorname{Ker} \tilde{\pi} = < N, N^{\varphi} > [N, G^{\varphi}] \cdot [G, N^{\varphi}]$.

Proof. (i) For elements $x \in N$ and $g, h \in G$, it follows that

$$\begin{aligned}[x, g^{\varphi}]^h &= [x, g^{\varphi}][x, g^{\varphi}, h] \\ &= [x, g^{\varphi}][x, g, h^{\varphi}] \quad \text{(by Lemma 2.1).}\end{aligned}$$

This implies that G normalizes $[N, G^{\varphi}]$, and similarly G^{φ} normalizes $[N, G^{\varphi}]$, from what we get $[N, G^{\varphi}] \trianglelefteq \mathcal{V}(G)$. An analogous argument shows that $[G, N^{\varphi}] \trianglelefteq \mathcal{V}(G)$.

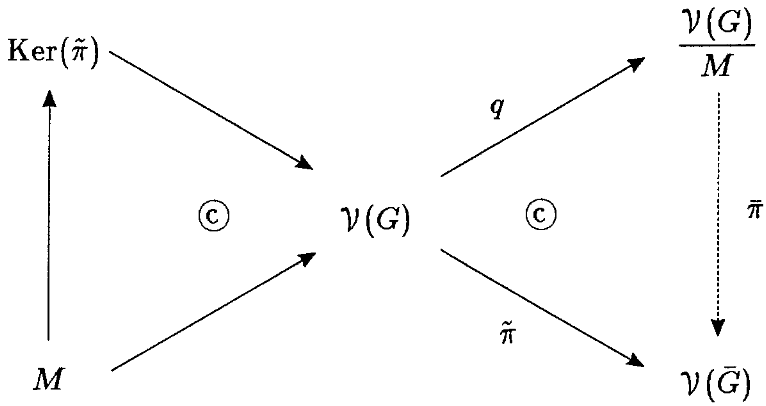
To prove (ii) we set $M = < N, N^{\varphi} > \cdot [N, G^{\varphi}] \cdot [G, N^{\varphi}]$, so that $M \leq \operatorname{Ker} \tilde{\pi}$. Furthermore M is a normal subgroup of $\mathcal{V}(G)$; thus we can define a function $\theta: \bar{G} \cup \bar{G}^{\varphi} \to \dfrac{\mathcal{V}(G)}{M}$ by setting $(\bar{g})\theta = Mg$ and $(\bar{g}^{\varphi})\theta = Mg^{\varphi}$, which is well defined since $N, N^{\varphi} \subseteq M$. The restrictions of θ to $\bar{G}$ and $\bar{G}^{\varphi}$ are both homomorphisms, so that there is a unique homorphism θ^* which extends θ to the free product $\bar{G} * \bar{G}^{\varphi}$. We see that the relations

$$[\bar{g}_1 \bar{g}_2, \bar{g}_3^{\varphi}] = [\overline{(g_1^{g_2})}, \overline{(g_3^{g_2})}^{\varphi}][\bar{g}_2, \bar{g}_3^{\varphi}]$$

and

$$[\bar{g}_1, (\bar{g}_2\bar{g}_3)^{\varphi}] = [\bar{g}_1, \bar{g}_3^{\varphi}][\overline{(g_1^{g_3})}, \overline{(g_2^{g_3})}^{\varphi}]$$

are preserved by θ^*. Consequently, θ induces a homomorphism $\tilde{\theta}\colon \mathcal{V}(\bar{G}) \to \frac{\mathcal{V}(G)}{M}$. Since $M \leq \mathrm{Ker}(\tilde{\pi})$ this yields an epimorphism $\bar{\pi}\colon \frac{\mathcal{V}(G)}{M} \to \mathcal{V}(\bar{G})$



such that $(Mg)\bar{\pi} = \bar{g}$ and $(Mg^{\varphi})\bar{\pi} = \bar{g}^{\varphi}$. By composition of $\tilde{\theta}$ and $\bar{\pi}$ we get $(\bar{g})\tilde{\theta}\bar{\pi} = \bar{g}$ and $(\bar{g}^{\varphi})\tilde{\theta}\bar{\pi} = \bar{g}^{\varphi}$, $\forall g \in G$. Thus $\tilde{\theta}\bar{\pi} = 1_{\mathcal{V}(\bar{G})}$, and this in turn shows that $\tilde{\theta}$ is an isomorphism. □

Now we want to consider the subgroup

$$\Upsilon(G) = [G, G^{\varphi}]$$

which is normal in $\mathcal{V}(G)$.

By the early definition of the non-abelian tensor square $G \otimes G$ we see that the map $\tau\colon G \otimes G \to \Upsilon(G)$ defined on the generators by $(g_1 \otimes g_2)^{\tau} = [g_1, g_2^{\varphi}]$ extends to an epimorphism from $G \otimes G$ to $\Upsilon(G)$. In fact we have

2.6 Proposition. *τ is an isomorphism.*

Proof. Firstly we look at the free product $G * G^{\varphi}$. Its subgroup $[G, G^{\varphi}]$ is free, freely generated by the commutators $[g_1, g_2^{\varphi}]$ where $1 \neq g_1 \in G, 1 \neq g_2^{\varphi} \in G^{\varphi}$. (See for instance [7], chap. 4). As a normal subgroup of $G * G^{\varphi}$, $[G, G^{\varphi}]$ admits the actions of G and G^φ by conjugation and the following identities hold

$$(I)\begin{cases} [g_1, g_2^\varphi]^g = [g_1 g, g_2^\varphi][g, g_2^\varphi]^{-1} \\ [g_1, g_2^\varphi]^{g^\varphi} = [g_1, g^\varphi]^{-1} \cdot [g_1, (g_2 g)^\varphi], \end{cases}$$

for all $g, g_1, g_2 \in G$.

Now the map $\mu: [G, G^\varphi] \to G \otimes G$ defined on the free generator $[g_1, g_2^\varphi]$ by $[g_1, g_2^\varphi]^\mu = g_1 \otimes g_2$ extends to an epimorphism from the (free) group $[G, G^\varphi]$ ($\trianglelefteq G * G^\varphi$) onto $G \otimes G$. Consequently, the introduction in $G * G^\varphi$ of the defining relations of $\mathcal{V}(G)$ takes us to describe $\Upsilon(G)$ as the quotient of $[G, G^\varphi]$ (still a subgroup of $G * G^\varphi$) by the relations

$$(II)\begin{cases} [g_1 g_2, g_3^\varphi] = [g_1^{g_2}, (g_3^{g_2})^\varphi][g_2, g_3^\varphi] \\ [g_1, (g_2 g_3)^\varphi] = [g_1, g_3^\varphi] \cdot [g_1^{g_3}, (g_2^{g_3})^\varphi], \end{cases}$$

for all $g_1, g_2, g_3 \in G$. But relations (II) are mapped by μ in the defining relations of $G \otimes G$, from what we get that μ induces an epimorphism from $\Upsilon(G)$ onto $G \otimes G$. We have $\mu\tau = 1_{\Upsilon(G)}$ and $\tau\mu = 1_{G\otimes G}$, thus proving our assertion. □

Remark 3. An argument similar to that used in Proposition 2.5 (ii) may be used to show if N is a normal subgroup of G and $\tilde{\pi}: \mathcal{V}(G) \to \mathcal{V}\left(\frac{G}{N}\right)$ is the epimorphism induced by the projection $\pi: G \to \frac{G}{N}$, then $\mathrm{Ker}(\tilde{\pi}) \cap \Upsilon(G) = [N, G^\varphi] \cdot [G, N^\varphi]$.

We close this section by proving

2.7 Proposition. *Let*

$$G = G_1 \trianglerighteq G_2 (= G') \trianglerighteq \cdots \trianglerighteq G_j \trianglerighteq \cdots,$$

$$1 = \xi_0(G) \trianglelefteq \xi_1(G) (= Z(G)) \trianglelefteq \cdots \trianglelefteq \xi_j(G) \trianglelefteq \cdots,$$

and

$$G = \gamma_1(G) \trianglerighteq \gamma_2(G) \trianglerighteq \cdots \trianglerighteq \gamma_j(G) \trianglerighteq \cdots$$

be respectively the derived series, the upper central series and the lower central series of G. Then

(i) $[\xi_j(G), G_{j+1}^\varphi] = 1$, *for all* $j \geq 0$;

(ii) $[\xi_{j+1}(G), \gamma_j(G^\varphi)] \cdot [\gamma_j(G), \xi_{j+1}(G^\varphi)]$ *is central in* $\Upsilon(G)$ *for all* $j \geq 1$;

(iii) $[\xi_j(G), \gamma_j(G^\varphi)]$ *is central in* $\mathcal{V}(G)$, *for all* $j \geq 1$.

Proof. (i) is trivial for $j=0$ while the general case follows directly from Lemma 2.2, since $G_j \leq \gamma_j(G)$ and $[\xi_j(G),\gamma_j(G)]=1$ for all $j\geq 1$.

(ii) for $j\geq 1, z\in \xi_{j+1}(G), g\in\gamma_j(G)$ and $g_1,g_2\in G$ we have

$$\begin{aligned}
[[z,g^{\varphi}],[g_1,g_2^{\varphi}]] &= [z,g^{\varphi}]^{-1}[z,g^{\varphi}]^{[g_1,g_2^{\varphi}]} \\
&= [z,g^{\varphi}]^{-1}[z,g^{\varphi}]^{[g_1,g_2]} \quad \text{(Lemma 2.1 (i))} \\
&= [z,g^{\varphi},[g_1,g_2]] \\
&= [z,g,[g_1,g_2]^{\varphi}] \quad \text{(Lemma 2.1 (ii))} \\
&= 1 \quad \text{(by Lemma 2.2, since} [\xi_{j+1}(G),\gamma_j(G)] \leq \xi_1(G)).
\end{aligned}$$

This implies that $\Upsilon(G)$ centralizes $[\xi_{j+1}(G),\gamma_j(G^{\varphi})]$ and by symmetry $\Upsilon(G)$ also centralizes $[\gamma_j(G),\xi_{j+1}(G^{\varphi})]$.

(iii) This part follows directly from Lemma 2.1 (ii) since $[\xi_j(G),\gamma_j(G)]=1$, for all $j\geq 1$. □

3. The Main Results

The description of $\mathcal{V}(G)$ as the product $\mathcal{V}(G)=\Upsilon(G)\cdot G\cdot G^{\varphi}$, which comes from the fact that $\Upsilon(G)\trianglelefteq\mathcal{V}(G)$, gives an elegant description for the lower central series and the derived series of $\mathcal{V}(G)$.

3.1 Theorem. *For $i\geq 2$ the i-th term of the lower central series of $\mathcal{V}(G)$ is given by*

$$\gamma_i(\mathcal{V}(G))=\gamma_i(G)\gamma_i(G^{\varphi})[\gamma_{i-1}(G),G^{\varphi}][G,\gamma_{i-1}(G^{\varphi})]$$

Proof. For $i=2, \gamma_2(\mathcal{V}(G))=[\mathcal{V}(G),\mathcal{V}(G)]=[\Upsilon(G)\cdot G\cdot G^{\varphi},\Upsilon(G)\cdot G\cdot G^{\varphi}]$. By using the defining relations of $\mathcal{V}(G)$ together with Lemma 2.1 and Proposition 2.5 (i) we get

$$[\Upsilon(G)\cdot G\cdot G^{\varphi},\Upsilon(G)\cdot G\cdot G^{\varphi}]\leq\Upsilon(G)\cdot\gamma_2(G)\cdot\gamma_2(G^{\varphi}).$$

This shows that $\gamma_2(\mathcal{V}(G))=\gamma_2(G)\gamma_2(G^{\varphi})\cdot\Upsilon(G)$. Suppose, by induction on $i\geq 2$, that

$$\gamma_i(\mathcal{V}(G))\leq\gamma_i(G)\gamma_i(G^{\varphi})[\gamma_{i-1}(G),G^{\varphi}]\cdot[G,\gamma_{i-1}(G^{\varphi})].$$

Then by Proposition 2.5 (i),

$$[\gamma_i(\mathcal{V}(G)),G]\leq\gamma_{i+1}(G)\cdot[\gamma_i(G^{\varphi}),G]\cdot[\gamma_{i-1}(G),G^{\varphi},G]\cdot[G,\gamma_{i-1}(G^{\varphi}),G],$$

and once more invoking Lemma 2.1 (i) we obtain

$$[\gamma_{i-1}(G), G^{\varphi}, G] = [\gamma_i(G), G^{\varphi}] = [G, \gamma_{i-1}(G^{\varphi}), G].$$

Therefore $[\gamma_i(\mathcal{V}(G)), G] \leq \gamma_{i+1}(G) \cdot [\gamma_i(G), G^{\varphi}] \cdot [G, \gamma_i(G^{\varphi})]$. By symmetry it follows that

$$[\gamma_i(\mathcal{V}(G)), G^{\varphi}] \leq \gamma_{i+1}(G^{\varphi})[\gamma_i(G), G^{\varphi}][G, \gamma_i(G^{\varphi})],$$

and these last two inclusions show that

$$\gamma_{i+1}(\mathcal{V}(G)) \leq \gamma_{i+1}(G) \cdot \gamma_{i+1}(G^{\varphi})[\gamma_i(G), G^{\varphi}][G, \gamma_i(G^{\varphi})],$$

so that our theorem is proved by induction. □

3.2 Corollary. *Let G be a nilpotent group of class c. Then $\mathcal{V}(G)$ is a nilpotent group of class at most $c+1$.*

The next theorem is proved using, step by step, similar arguments as in the proof of Theorem 3.2. We will omit its proof.

3.3 Theorem. *For $i \geq 2$ the i-th term of the derived series of $\mathcal{V}(G)$ is given by*

$$\mathcal{V}(G)_i = G_i G_i^{\varphi}[G_{i-1}, G_{i-1}^{\varphi}],$$

where G_i, denotes the i-th term of the derived series of G.

3.4 Corollary. *Let G be a solvable group of derived length ℓ. Then $\mathcal{V}(G)$ is solvable of derived length at most $\ell+1$.*

3.5 Proposition. *Let $G = N \cdot H$ be a semidirect product of its subgroups $N \trianglelefteq G$ and $H \leq G$. Then*

(i) $\mathcal{V}(G) = \langle N, N^{\varphi}\rangle[N, H^{\varphi}][H, N^{\varphi}] \cdot \langle H, H^{\varphi}\rangle$;

(ii) $\langle H, H^{\varphi}\rangle \cong \mathcal{V}(H)$.

Proof. (i), (ii). It follows easily from Proposition 2.5 that $[N, H^{\varphi}]$ and $[H, N^{\varphi}]$ are both normal subgroups of $\mathcal{V}(G)$; also, $\langle N, N^{\varphi}\rangle[N, H^{\varphi}][H, N^{\varphi}]$ is actually the Kernel of $\tilde{\pi}\colon \mathcal{V}(G) \to \mathcal{V}\left(\frac{G}{N}\right)$ $(\cong \mathcal{V}(H))$. On writing $\mathcal{V}(G) = \mathcal{V}(NH) = [NH, N^{\varphi}H^{\varphi}] \cdot NH \cdot N^{\varphi}H^{\varphi}$ we see that

$$[NH, N^{\varphi}H^{\varphi}] \leq [N, N^{\varphi}][N, H^{\varphi}][H, N^{\varphi}]$$

and thus $\mathcal{V}(G)$ has the desired expression. As for (ii), $\langle H, H^{\varphi}\rangle^{\tilde{\pi}} = \mathcal{V}\left(\frac{G}{N}\right)$ ($\cong$ $\mathcal{V}(H)$), while on the other hand $\mathcal{V}(H)$ is mapped onto $\langle H, H^{\varphi}\rangle$. Therefore $\mathrm{Ker}(\tilde{\pi}) \cap \langle H, H^{\varphi}\rangle = \{1\}$ and $< H, H^{\varphi} > \cong \mathcal{V}(H)$. □

3.6 Proposition. *Let $G = N \times H$ be the direct product of its normal subgroups N and H. Then*

(i) $\mathcal{V}(G) = \langle N, N^{\varphi}\rangle \cdot [N, H^{\varphi}] \cdot [H, N^{\varphi}] \cdot < H, H^{\varphi} >$

(ii) $\langle N, N^{\varphi}\rangle \cong \mathcal{V}(N)$; $\quad \langle H, H^{\varphi}\rangle \cong \mathcal{V}(H)$

(iii) $\Upsilon(G) = \Upsilon(N) \times \Upsilon(H)$.

Proof. Parts (i) and (ii) follows from double application of Proposition 3.5. As for (iii), we get from Proposition 2.7 (i) that the four subgroups $[N, H^{\varphi}], [N, N^{\varphi}]$, $[H, N^{\varphi}]$ and $[H, H^{\varphi}]$ are mutually centralized in $\Upsilon(G)$, since $[N, H] = 1$. Also, normality of $[N, H^{\varphi}]$ and $[H, N^{\varphi}]$ in $\mathcal{V}(G)$ give

$$\Upsilon(G) = [N, N^{\varphi}] \cdot [N, H^{\varphi}][H, N^{\varphi}][H, H^{\varphi}].$$

Lastly we observe that part (ii) implies $[N, N^{\varphi}] \cong \Upsilon(N)$ and $[H, H^{\varphi}] \cong \Upsilon(H)$. □

Remark 4. The result in Part (iii) is Proposition 11 of [1].

In fact, by arguments similar to those used in Proposition 2.6 we can prove that when H and K are groups which act trivially on each other (but by conjugation on themselves) then the subgroup $[H, K^{\varphi}]$ of $\mathcal{V}(H \times K)$ is isomorphic to $H \otimes K$ which in turn is the usual tensor product $H \otimes_{\mathbb{Z}} K$ (this follows from Lemma 2.1; see also Remark 2 of [1]).

Remark 5. In case of abelian groups A and B we have therefore the known decomposition of the ordinary tensor product: $(A \times B) \otimes_{\mathbb{Z}} (A \times B) \cong \Upsilon(A \times B) \cong (A \otimes_{\mathbb{Z}} A) \times (A \otimes_{\mathbb{Z}} B) \times (B \otimes_{\mathbb{Z}} A) \times (B \otimes_{\mathbb{Z}} B)$.

3.7 Corollary. *Let $G = P_1 \times \cdots \times P_n$ be a finite nilpotent group where $\{P_1, \cdots, P_n\}$ is the set of distinct Sylow p-subgroups of G. Then,*

(i) $\mathcal{V}(G) \cong \mathcal{V}(P_1) \times \cdots \times \mathcal{V}(P_n)$

(ii) $\Upsilon(G) \cong \Upsilon(P_1) \times \ldots \times \Upsilon(P_n)$

Proof. For any prime p dividing $|G|$, let P be a Sylow p-subgroup of G and N be a normal p-complement in G. We have by Lemma 2.3 (ii) that $[N, P^{\varphi}] = [P, N^{\varphi}] = 1$.

The previous proposition then yields $\mathcal{V}(G) \cong \mathcal{V}(N) \times \mathcal{V}(P)$ and $\Upsilon(G) \cong \Upsilon(N) \times \Upsilon(P)$. Parts (i), (ii) now follow by induction on $n \geq 2$. □

From now on we restrict ourselves to the case of a finite p-group G.

3.8 Lemma. *Let G be a finite p-group and $c \in Z(G) \cap G'$ be an element of order p. If $\phi(G)$ denotes the Frattini subgroup of G, then*

$$|\mathcal{V}(G)| \quad \textit{divides} \quad p^2 \left|\frac{G}{\phi(G)}\right| \left|\mathcal{V}\left(\frac{G}{< c >}\right)\right|$$

Proof. By Proposition 2.7 (i) we get $[c, g^{\varphi}] = 1$ for all $g \in G'$. On the other hand, if $x \in G$ then, by Lemma 2.3 (i), $[c, (x^p)^{\varphi}] = [c, x^{\varphi}]^p = [c^p, x^{\varphi}] = 1$, so that $[c, g^{\varphi}] = 1$ for all $g \in G^p := \langle x^p | x \in G \rangle$. It follows that $[c, \phi(G)^{\varphi}] = 1$ since $\phi(G) = G'G^p$. If we set $\lambda: G \rightarrow [c, G^{\varphi}], g \mapsto [c, g^{\varphi}]$ then λ is an epimorphism, as $[c, G^{\varphi}]$ is central in $\mathcal{V}(G)$. Also, $\phi(G) \leq \text{Ker}(\lambda)$. Let $\pi: G \rightarrow \frac{G}{< c >}$ be the canonical projection and $\tilde{\pi}$ its induced in $\mathcal{V}(G)$, whose kernel is $\text{Ker}(\tilde{\pi}) =< c >< c^{\varphi} > [c, G^{\varphi}][G, c^{\varphi}]$. Let $\bar{a}$ be a generator of $\frac{G}{\phi(G)}$. If c is a simple commutator, say $c = [x, y]$, then we get

$$\begin{aligned} [a, c^{\varphi}] &= [a, [x, y]^{\varphi}] \\ &= [[x, y]^{\varphi}, a]^{-1} \\ &= [x, y^{\varphi}, a]^{-1} \quad \text{(by Lemma 2.1 (ii))} \\ &= [a, [x, y^{\varphi}]]. \end{aligned}$$

In general, if c is a product of commutators, say $c = [x_1, y_1][x_2, y_2] \dots [x_r, y_r]$, then by induction we get $[a, c^{\varphi}] = [a, [x_1, y_1^{\varphi}] \dots [x_r, y_r^{\varphi}]]$.

Analogously, $[c, a^{\varphi}] = [[x_1, y_1^{\varphi}] \dots [x_r, y_r^{\varphi}], a]$. Since $c \in Z(G) \cap G'$, it follows from the above identities that, in $[c, G^{\varphi}][G, c^{\varphi}]$, the elements of the form $[c, a^{\varphi}][a, c^{\varphi}]$ are all trivial.

On the other hand if $\frac{G}{\phi(G)}$ is generated by $\{\bar{a}_1, \dots, \bar{a}_d\}$, we have

$$[c, (a_1^{i_1} \dots a_d^{i_d})^{\varphi}] = [c, a_1^{\varphi}]^{i_1} \dots [c, a_d^{\varphi}]^{i_d}$$

and

$$[(a_1^{j_1}\dots a_d^{j_d}),c^\varphi]=[a_1,c^\varphi]^{j_1}\dots[a_d,c^\varphi]^{j_d},$$

by Lemma 2.3 (i). This means that $[c,G^\varphi][G,c^\varphi]$ is generated by the $2d$ elements

$$[a_1,c^\varphi],\dots,[a_d,c^\varphi],[c,a_1^\varphi],\dots,[c,a_d^\varphi].$$

But since $[a_i,c^\varphi][c,a_i^\varphi]=1$ for $i=1,\dots,d$, it results that $[c,G^\varphi][G,c^\varphi]=[c,G^\varphi]$, which is generated by $[c,a_1^\varphi],\dots,[c,a_d^\varphi]$. This together with the fact that λ is an epimorphism gives

$$|\mathrm{Ker}(\tilde\pi)|\le p^2\cdot|[c,G^\varphi]|\le p^2\cdot\left|\frac{G}{\phi(G)}\right|.$$

Therefore $|\mathcal{V}(G)|$ divides $p^2\left|\frac{G}{\phi(G)}\right|\cdot\left|\mathcal{V}\left(\frac{G}{<c>}\right)\right|$. □

3.9 Proposition. *Let G be a finite p-group of class 2. Then $|\Upsilon(G)|$ divides*

$$\left|G'\otimes_{\mathbb{Z}}\frac{G}{G'}\right|\cdot\left|\Upsilon\left(\frac{G}{G'}\right)\right|.$$

Proof. Let $\tilde\pi\colon\mathcal{V}(G)\to\mathcal{V}\left(\frac{G}{G'}\right)$ be the epimorphism induced by the canonical map $\pi\colon G\to\frac{G}{G'}$. By Remark 3 we have $\mathrm{Ker}(\tilde\pi)\cap\Upsilon(G)=[G',G^\varphi][G,(G')^\varphi]$, while $\Upsilon(G)^{\tilde\pi}=\Upsilon\left(\frac{G}{G'}\right)$. Thus it remains to evaluate $|\mathrm{Ker}(\tilde\pi)\cap\Upsilon(G)|$. Since $G'\le Z(G)$, Proposition 2.7 (i) gives $[G',G'^\varphi]=1$. Hence, for $c\in G'$ and $g=dh\in G$, where $h\in G'$, $[c,(dh)^\varphi]=[c,h^\varphi][c,d^\varphi]^{h^\varphi}=[c,d^\varphi]$.

As $[G',G^\varphi]$ is central in $\mathcal{V}(G)$ (Proposition 2.7 (iii)), this implies that $[G',G^\varphi]$ is a homomorphic image of $G'\otimes_{\mathbb{Z}}\frac{G}{G'}$ through the map $c\otimes\bar d\mapsto[c,d^\varphi]$, where $c\in G'$ e $\bar d=d^\pi$.

Therefore $|[G',G^\varphi]|$ divides $\left|G'\otimes_{\mathbb{Z}}\frac{G}{G'}\right|$. Suppose $G'=\langle c_1,\dots,c_m\rangle$ and $\frac{G}{G'}=\langle\bar d_1,\dots,\bar d_n\rangle$. Then $[G',G^\varphi]$ is generated by the set $\{[c_i,d_j^\varphi],1\le i\le m,1\le j\le n\}$ and similarly $[G,(G')^\varphi]$ is generated by $\{[d_j,c_i^\varphi],1\le j\le n,1\le i\le m\}$. But each c_i is a product of commutators so that we get, as in the proof of Lemma 3.8, $[c_i,d_j^\varphi][d_j,c_i^\varphi]=1$, for all pairs (i,j). This in turn gives $[G',G^\varphi][G,(G')^\varphi]=[G',G^\varphi]$, and consequently $|\mathrm{Ker}(\tilde\pi)\cap\Upsilon(G)|$ divides $\left|G'\otimes\frac{G}{G'}\right|$. □

3.10 Corollary. *Let G be a p-group of class ≤ 2 with $|G| = p^n$ and $|G'| = p^m$. Then $|\Upsilon(G)|$ divides $p^{n(n-m)}$.*

Proof. We observe that $\left|G' \otimes_{\mathbb{Z}} \frac{G}{G'}\right|$ divides $p^{m(n-m)}$ and

$$\left|\Upsilon\left(\frac{G}{G'}\right)\right| = \left|\frac{G}{G'} \otimes_{\mathbb{Z}} \frac{G}{G'}\right|$$

divides $p^{(n-m)^2}$. □

3.11 Theorem. *Let G be a finite p-group with $|G| = p^n$ and $|G'| = p^m$. Then $|\mathcal{V}(G)|$ divides $p^{n^2+2n-mn}$.*

Proof. Since $|\mathcal{V}(G)| = |\Upsilon(G)| \cdot |G|^2$, all we need is to evaluate $|\Upsilon(G)|$. If G has nilpotence class ≤ 2 then we are done with Corollary 3.8.

Suppose G has class at least 3 and let $c \in \gamma_3(G) \cap Z(G)$ be an element of order p. We argument by induction on $|G|$. Since

$$\left|\frac{G}{< c >}\right| = p^{n-1} \quad \text{and} \quad \left|\left(\frac{G}{< c >}\right)'\right| = p^{m-1},$$

our hypothesis give that $\left|\Upsilon\left(\frac{G}{< c >}\right)\right|$ divides $p^{(n-1)(n-m)}$.

On the other hand $\left|\frac{G}{\phi(G)}\right|$ divides $\left|\frac{G}{G'}\right| = p^{n-m}$, so that by Lemma 3.8 we finally obtain $|\Upsilon(G)|$ divides $p^{n(n-m)}$. □

3.12 Corollary. *Let $|G| = p^n, |G'| = p^m$ and $d = d(G)$ be the minimal number of generators of G. Then*

$$p^{d^2} \leq |G \otimes G| \leq p^{n(n-m)}$$

Proof. We observe that on making $N = \phi(G)$ and

$$\tilde{\pi}: \mathcal{V}(G) \to \mathcal{V}\left(\frac{G}{\phi(G)}\right)$$

in Proposition 2.5, then by Remark 3 it results that

$$\mathrm{Ker}(\tilde{\pi}) \cap \Upsilon(G) = [\phi(G), G^{\varphi}][G, \phi(G)^{\varphi}],$$

so that the restriction of $\tilde{\pi}$ to $\Upsilon(G)$ renders

$$|\Upsilon(G)| \geq \left|\Upsilon\left(\frac{G}{\phi(G)}\right)\right| = \left|\left[\frac{G}{\phi(G)}, \left(\frac{G}{\phi(G)}\right)^{\varphi}\right]\right|.$$

But $\dfrac{G}{\phi(G)}$ is elementary abelian of order p^d and (as observed in Remarks 4 and 5.)

$$\left[\frac{G}{\phi(G)}, \left(\frac{G}{\phi(G)}\right)^{\varphi}\right]$$

is precisely the usual tensor product

$$\frac{G}{\phi(G)} \otimes_{\mathbb{Z}} \frac{G}{\phi(G)},$$

of order p^{d^2}.

On the other hand the last theorem gives the upper bound

$$|\Upsilon(G)| = \frac{|\mathcal{V}(G)|}{|G|^2} \leq p^{n^2-mn}.$$

Our proof is now finished by the isomorphism of Proposition 2.6. □

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